Effects of noise coherence on stochastic resonance enhancement in a bithreshold system

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We identify a method for optimizing the stochastic resonance (SR) in a symmetric bithreshold device: by varying the coherence of the added noise series. To show SR enhancement via this method, we compare the performance of the system using noise sources with different coherence at normalized amplitude. The normalization of the noise amplitude is based on the mean threshold crossing rate of the Gaussian white noise, which is considered as the standard noise in SR studies, at optimal variance. The amplitude for optimal performance of the Gaussian white noise is determined using a signal-to-noise ratio (Q). The Q measure is also used to compare and examine the system performance for different noise cases. This measure is used because it is particularly sensitive to the effects of coherence on the quality of the output power spectrum.

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I. INTRODUCTION

Noise has dual nature. It usually degrades the performance of linear systems but it has also been shown to enhance the sensitivity of other nonlinear systems to weak signals [1]. Although counterintuitive, the beneficial aspect of noise is now established as stochastic resonance (SR) [2,3]. In SR, the system performance increases with increasing noise amplitude up to an optimal value. Thereafter, performance declines as is usually expected when the noise in a system increases.

In a system that exhibits SR, the performance is sensitive to the noise characteristics. Standard SR is formulated and characterized with a Gaussian white noise as the added noise source [4–9]. Real noise, however, is usually colored and has intrinsic coherence that could affect SR behavior. To be able to properly evaluate SR performance, coherence effects should be taken into account.

Several studies dealing with multistable systems claim that the coherence in an added perturbation is responsible for some unexpected behaviors in the system. Among these are the breaking of dynamical parity of a particle in a symmetric periodic potential [10] and the enhancement of performance of a neural network that is applied to an optimization task [11]. In other work [12], the performance of a sinusoidal threshold system that is used to detect a weak signal is enhanced when noise is added to induce threshold crossings. This system is similar to a symmetric bithreshold device that uses a noisy sinusoid to force the weak signal across the thresholds. This suggests that coherence in an added perturbation can improve the detection in a similar system.

The effect of noise color on SR, particularly for threshold systems, has been considered in some studies [13–18]. These works, however, did not normalize the noise amplitude. For the present study, we normalize the amplitude of different types of noise added so as to ‘‘decouple’’ the amplitude effects from the coherence effects. We then examine the effect of the coherence on weak signal detection using a bithreshold device. We show that noise coherence does improve SR performance.

In Sec. II, we introduce the elements of the detection procedure under study. This section includes the description of the system, the sampling parameters, the weak sinusoid, the added noise, and the computation of the power spectrum used in the numerical simulations. In Sec. III, we state the normalization condition imposed on the noise amplitude. In Sec. IV, we discuss the measure of performance used to evaluate SR. We present the results of the numerical experiments in Sec. V followed by a discussion in Sec. VI. Lastly, we summarize the results of this work in Sec. VII.

II. ELEMENTS OF THE DETECTION PROCEDURE

A. The detector and the sampling parameters

The system under study is a symmetric bithreshold trigger device described by

$$y=\begin{cases} +1, & x\geqslant B \\ 0, & -B<x<B \\ -1, & x\leqslant -B, \end{cases} \tag{1}$$

where $B=0.5$ arbitrary units. This is the simplest symmetric system that exhibits SR [17]. It represents a class of threshold devices that includes the experimental Schmitt trigger, which is a similar system albeit exhibiting hysteresis [15]. The simplicity of this model allows comparison with a wide class of systems.

In simulating this system as a detector, an input signal is sampled at equal time intervals ΔT_{samp}. If the absolute value of the sample is equal to or greater than the threshold value $|B|$, then we say that the threshold was crossed. Threshold crossing events or trigger events are time marked with a pulse of amplitude $+1$ or -1, depending on which threshold is crossed. Between trigger events, the system output is null.

In the numerical experiments, the sampling parameters are arbitrarily set to frequency $f_{samp}=800$ Hz for a total of 1600 samples. The corresponding sampling period $\Delta T_{samp}=0.001\,25$ s is smaller by at least an order of magnitude

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compared to the other time scales of interest (i.e., the period of the input signal and the relevant noise correlation time).

B. The input signal

The input $x[n]$ to the detector is composed of the weak periodic modulation plus the noise term as follows:

$$x[n]=A_s \cos[2\Pi f_s(n\Delta T_{samp})]+\xi^n, \tag{2}$$

where $f_s=100$ Hz is the signal frequency and $A_s=0.3$ arb. units is the signal amplitude. Note that $A_s<|B|$ so that the sinusoid alone cannot surpass the threshold, and is thus undetectable.

The term ξ^n represents the noise added to the subthreshold sinusoid necessary to induce threshold crossings. To study the effects of noise coherence on SR enhancement, several cases of the noise term ξ^n are considered. The normalization of the noise amplitude, which is defined as the standard deviation, will be discussed later in the paper.

(1) (*Gaussian white noise*) (ξ_{gwn}). The Gaussian distributed noise source is generated using the Box-Muller procedure [19]. The tuning parameter is D(arb. units), which is defined as the standard deviation of the Gaussian noise.

(2) *Periodic noise* (ξ_{pn}). We investigate for completeness the effect of infinite correlation on SR performance. As a noise source, we use another periodic signal with suprathreshold amplitude given by

$$\xi_{pn}^{n+1}=A_{pn}\cos[2\Pi f_{pn}(n\Delta T_{samp})] \tag{3}$$

where $A_{pn}\geqslant|B|$ is the amplitude and $f_{pn}\neq f_s$ is the frequency, which serves as the tuning parameter.

(3) *Time-correlated noise* (ξ_{tcn}) and its *shuffled* version (ξ_{stcn}). The exponentially correlated noise is calculated based on the following Ornstein-Uhlenbeck equation [20]:

$$\dot{\xi}_{tcn}=-\frac{\xi_{tcn}}{T_c}+\frac{\xi_{Gwn}}{T_c} \tag{4}$$

where T_c is associated with the correlated time of the series and is tunable from $(0,\infty)$ and ξ_{Gwn} is Gaussian white noise with zero mean and unit variance. ξ_{tcn} has the following properties:

$$\langle\xi_{tcn}(t)\rangle=0, \tag{5}$$

$$\langle\xi_{tcn}(t)\xi_{tcn}(s)\rangle=\sigma^2\exp-\frac{|t-s|}{T_c} \tag{6}$$

where $\langle\cdots\rangle$ signifies averaging and σ is the standard deviation.

For comparison, the same series is used but with the coherence removed. To remove the correlation in ξ_{tcn}, the entire sequence is shuffled before it is added to the weak periodic signal. Shuffling removes only the correlation but maintains the distribution and amplitude of its unshuffled counterpart.

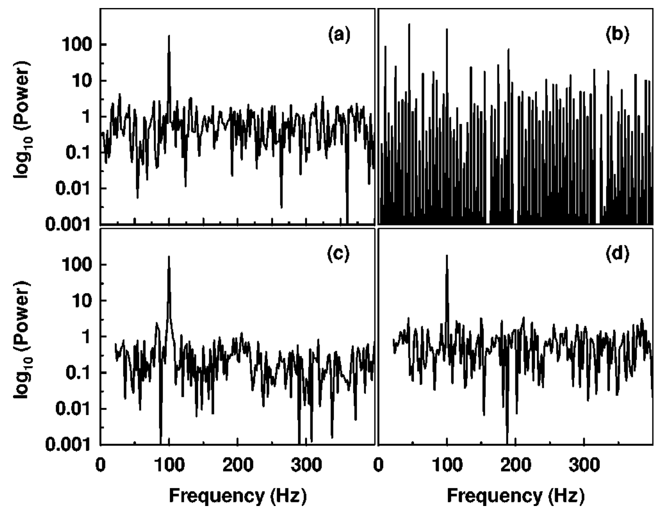

FIG. 1. Power spectra obtained using (a) ξ_{Gwn} at $D=0.32$; (b) ξ_{pn} at $f_{pn}=45$ Hz; (c) ξ_{tcn} and (d) ξ_{stcn} at $T_c=0.01$ s. The true signal frequency is prominent at 100 Hz, except in (b).

C. Computation of the power spectrum

The frequency of the sinusoid is determined from the power spectrum computed from the output $\{t_n, y_n\}$ of the system. The usual methods of computing the power spectrum require samples obtained at equal time intervals. The random nature of the threshold detection problem, however, dictates that crossing events need not happen at equal time intervals. To compute the unequal time interval samples, we use the Lomb spectrogram [21].

We now examine the best spectrum obtained for each noise case as shown in Fig. 1. Except for the spectrum for periodic noise [see Fig. 1(b)], the true frequency at $f=100$ Hz is easily distinguishable from all the other components in the spectra shown. These spectra show that the crossings induced by the added noise are not purely random and that they hold information about the periodicity of the subthreshold signal. The prominent peak at the true signal frequency is evidence of such entrainment.

III. NORMALIZATION CONDITION

Correlation describes a temporal feature of noise that is distinct from the information gained from the distribution or statistics (e.g., mean and amplitude) of the noise. But because it characterizes the similarity or agreement of noise values over time, it is therefore not independent of noise amplitude *per se*. When comparing the correlation effects of different noise types on SR behavior, there is a need to first decouple the correlation from the amplitude. This allows for a comparison of the performance of the system under different noise cases so that the enhancement or degradation that is due mainly to the correlation can be pointed out. We suggest that the amplitude of the noise be normalized. Normalization has not been considered in other studies.

A suitable normalizing condition not only standardizes the noise amplitude but also reflects the interaction between the system and the noise. Normalizing the mean threshold crossing rate of the noise source satisfies these prerequisites since

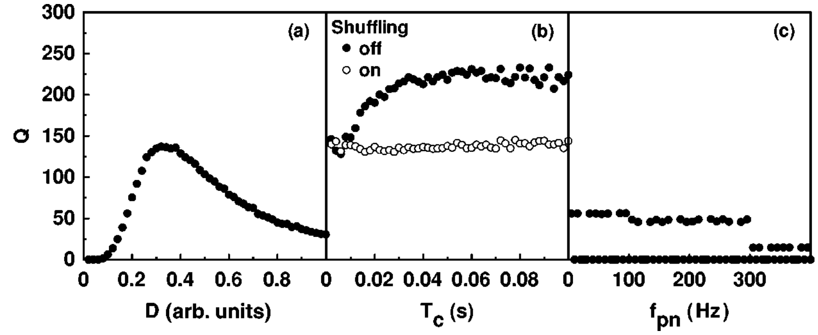

FIG. 2. Plots of Q against tuning parameters for different cases of correlation. For 0 correlation (ξ_{Gwn}) (a), the tuning parameter is D (arb. units). For finite correlation (ξ_{tcn}) (b), the tuning parameter is the correlation time of series T_c. For infinite correlation (ξ_{pn}) in (c), the tuning parameter is the frequency f_{pn} of the added periodic signal. For the periodic case, the input signal is periodic so that the normalization condition is satisfied only at certain frequencies, hence the gaps in the plot. Except for the periodic case, results are obtained over 100 runs.

it reflects on the amplitude of the noise and it is defined relative to the system threshold. To preset the standard value ξ_{Gwn}, which is typically used in SR studies, is applied to the system. The Q performance curve is obtained and the optimal noise amplitude is determined. Next, Gaussian white noise at the optimal amplitude is applied to the threshold system and the mean number of crossing events N within the observation time (threshold crossing rate) is determined. In implementing this procedure, the ξ_{Gwn} at optimal amplitude triggers on average 225 crossings or one crossing every seven sampling points. Once N is known, all the other noise sources are first scaled (amplitude axis) to satisfy this N threshold crossing condition before they are added to the weak signal.

IV. MEASURE OF PERFORMANCE

The SR performance is evaluated by how much the true frequency rises above the noise background in the power spectrum. Before the performance can be evaluated, the true frequency must first be identified. This is done by comparing samples of the power spectra over the respective tuned parameters for each noise case. The frequency that persists prominently in the spectra over the range of tuned parameters is taken as the true frequency.

In the periodic noise case, two frequencies naturally persist, one of them being the added frequency, which is known upon addition. The true frequency is taken as the other persistent component in the spectrum. In the time-correlated case, the spectrum is high-pass filtered to remove a band of low-frequency components that persist in the spectra but bear no relation to the true component. For the two other noise cases, the procedure establishes the true frequency without any further spectrum conditioning.

Performance is quantified based on how distinguishable the true frequency is from the other components in the spectrum. This measure Q is defined as

$$Q=\frac{P_{f_s}}{P_{var}} \tag{7}$$

where P_{f_s} is the power of the signal frequency f_s above the mean spectral power and P_{var} is the variance of the power over the frequencies. The ratio is used because it measures both the efficiency of the detection of the sinusoid, which is quantified by P_{f_s}, and the quality of the spectrum, which is quantified by P_{var}.

V. NUMERICAL RESULTS

The SR performances based on the Q measure for the different noise sources are shown in Fig. 2. The best Q value is obtained with ξ_{tcn}. Removing this correlation by shuffling the series or making the correlation time approach zero decreases the Q to values that are comparable with that of the white noise sources. These results clarify the role of temporal coherence in enhancing SR performance. As expected, the worst Q value is obtained with ξ_{pn}. This reflects the sensitivity of the Q measure to the presence of unwanted peaks in the spectrum, which is most evident in the spectrum for the periodic noise case. Originally, the concept of SR was about the existence of an optimal noise amplitude at which performance is maximized. In this work, we claim that SR is also about the existence of a critical value of the noise correlation for enhanced performance of the detector.

VI. DISCUSSION

A. Effects of noise coherence

The influence of coherence on how the crossing events occur is understood by noting that within a characteristic time length the noise values would normally have the same sign or polarity. Therefore, when a positive run of these values is superimposed on the sinusoid so that it crosses the threshold, the detector state is biased to crossing the threshold within a correlation time length, resulting in more crossings than when an uncorrelated noise is used. These crossing events are well synchronized with the true sinusoid so that they are able to mark each peak of the true signal with a pulse of the right sign. If the correlation time of the noise T_c is comparable to the half period $1/2f_s$ of the signal, more crossings of the proper polarity will be induced.

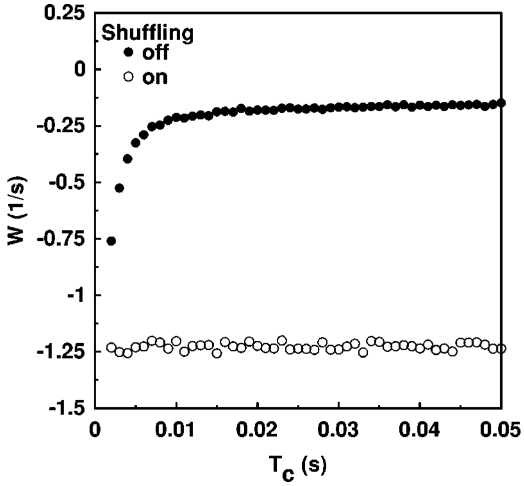

FIG. 3. Damping factor W for $\exp(Wt)$-fitted histogram of sequential noise values with the same sign. These results are obtained for the normalized ξ_{tcn} series. T_c is the correlation time of the series.

Another property of pulse crossings induced by time-correlated series is that the induced pulses occurs randomly. When random, the appearance of unwanted components in the power spectrum is suppressed, thereby enhancing the quality of the power spectrum. If the pattern of the crossings is strictly periodic, as in the case of periodic noise, spurious components are significantly enhanced.

B. Quantifying the coherence effects

To study the connection of the Q measure to the noise coherence, we analyze the lengths of time when the noise source is able to run values of the same polarity. We then compute the distribution of the such time lengths and calculate the damping factor W, assuming that the distribution is of the form $\exp(Wt)$. A higher (less negative) W value means a higher correlation in the noise series. The factor $(1/|W|)$ indicates the typical time length that supports pulse crossings of the same polarity. It can be considered as the numerical analog of the correlation time that is computed directly from the noise series.

Figure 3 shows the result of this computation for the time-correlated noise series. Qualitatively, the trend of the damping factor W curve is similar to that of the Q curve, both exhibiting saturation although at different slopes at the shoulder. Such differences can be attributed to the thresholding effect, which was not considered in the computation of W. This result suggests that the Q measure is sensitive to the damping factor W and hence to the coherence contained in the time-correlated noise series.

The observation that correlation enhances SR performance is contrary to a previous result dealing with an overdamped particle in a periodically forced double-well potential system [13,18], which has one unstable and two stable states. In this system, hopping of the particle between the two stable states is induced upon addition of noise. Synchronization of this hopping event with the occurrence of the peak in the sinusoid is necessary to obtain information about the sinusoid. For this particular system, however, the particle has only a single chance to hop and synchronize with the sinusoid within its half period. If the noise is correlated, the hoppings will be suppressed exponentially. This negatively affects the number of hoppings induced and in turn the synchronization with the sinusoid. Hence, there is a decrease in transmission of information from the sinusoidal input to the system output.

Hopping events are analogous to the crossing of a threshold for the bithreshold system. For the bithreshold system, threshold crossings are not limited to single occurrences per half period of the signal, so that synchronization is easier to achieve. To yield more crossings, it is good for the noise to maintain polarity within a time interval comparable to the half period of the sinusoid. This requirement is readily satisfied by a correlated noise. As expected, higher performance is obtained with a correlated noise than with a white noise.

VII. CONCLUSION

In this work, we investigated the role of correlation in the enhancement of SR performance. We showed that SR enhancement due to an optimal correlation in the added noise relies on proper matching between the half period of the sinusoid and the correlation time of the noise. This allows for more threshold crossings compared to the Gaussian white noise case. Aside from this, the random nature of the noise suppresses the appearance of unwanted harmonics in the spectrum, which is unavoidable for the periodic noise case.

A theoretical framework incorporating the results obtained is beyond the scope of this work. It would be interesting to show whether these results apply to the human hearing process. The auditory system has been suggested to use the SR mechanism to enhance hearing sensitivity [22]. It is of interest whether this same system uses intrinsic correlation in natural sounds to further enhance weak signal reception.

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